Hence $R = \sqrt{r^2 + 2hr}$, where r is the maximum horizontal range. This is the maximum range for a gun in the fort. Likewise the maximum range for a gun on the ship is

$$R_1 = \sqrt{r^2 - 2hr}.$$

Therefore the area required is

$$\pi R^2 - \pi R_1^2 = 4\pi hr.$$

Also solved by MRS. H. E. TREFETHEN and H. C. FEEMSTER.

NUMBER THEORY.

185. Proposed by R. D. CARMICHAEL, Bloomington, Ind.

Obtain the complete solution of the equation $\phi(p^\alpha) = \phi(q^\beta)$ where ϕ denotes Euler's ϕ-function, p and q are unknown primes, and α and β are unknown integers.

SOLUTION BY L. C. MATHEWSON.

From the theory of numbers[1] we have

$$\begin{aligned} \phi(p^\alpha) &= p^\alpha\left(1 - \frac{1}{p}\right) = p^{\alpha-1}(p-1); \\ \phi(q^\beta) &= q^\beta\left(1 - \frac{1}{q}\right) = q^{\beta-1}(q-1). \end{aligned} \tag{1}$$

Obviously the given equation is always satisfied by $p = q$, if $\alpha = \beta$. We, accordingly, consider the case where the primes are not equal, or $p \neq q$.

From (1) we have

$$p^{\alpha-1}(p-1) = q^{\beta-1}(q-1). \tag{2}$$

Since $p^{\alpha-1}$ and $q^{\beta-1}$ can have no common factors, we see that (2) gives

$$\begin{aligned} p^{\alpha-1} &= q - 1 \\ q^{\beta-1} &= p - 1 \end{aligned} . \tag{3}$$

It is sufficient to consider only the first of these equations, since the second is of the same form and leads to no new general results or solutions. Now $\alpha = 1$ or $\alpha > 1$. From (3) written in the form $p^{\alpha-1} + 1 = q$, we see that if $\alpha = 1$, $q = 2$, the even prime: if $\alpha > 1$, p cannot be an odd prime (for this would make q composite), hence p can be only 2. Since we are now excluding $p = q$, and because from (3) we have seen that the results will be interchangeable, and since we now see that at least one of the primes here must be the even prime, we consider p odd and q even. We have then to discuss $\phi(p^\alpha) = \phi(q^\beta)$, where p is an odd prime, $\alpha = 1$, $q = 2$; i. e., $\phi(p) = \phi(2^\beta)$.

We have simply $p - 1 = 2^{\beta-1}$, or $p = 2^{\beta-1} + 1$. Here p is an odd prime of the form $2^m + 1$. It is easily seen that m cannot be an odd prime nor an integral multiple of an odd prime (for in these cases since the binomial form itself is the

[1] See Bachmann's Neuere Zahlentheorie, p. 24, or any elementary text on the theory of numbers.

sum of the same odd powers of two integers, it is factorable); m can thus be only an integral power of 2; as, 1, 2, 4, 8, ,$\cdots$. This means that $\beta = 2^n + 1$, where n must be such a positive integer that $p = 2^{2^n} + 1$ is a prime.

Summarizing, we have for solutions of the given equation $\phi(p^\alpha) = \phi(q^\beta)$:

Case (1), $p = q$. Any prime, even or odd, is a root, if $\alpha = \beta$.

Case (2), $p \neq q$. One root is the even prime 2 with any exponent 2^n such that $2^{2^n} + 1$ is a prime, this odd prime being the other root for this individual solution. In this case the exponent of the odd prime is unity, while that of the even prime is of the form 2^n, as just indicated.

It may be added that the general form for n such that $2^{2^n} + 1$ is always a prime is still undetermined. It is of interest to note that this form is one which Fermat conjectured was always a prime for positive integral values of n. While this is true for many values of n, as 0, 1, 2, 3, $\cdots$, the factors have been found for other values of n: thus $2^{2^n} + 1$ is composite for $n = 5, 6, 9, 11, 12, 18, 23, 36, 38$.[1] These numbers are too large to be handled readily (or at all) when expressed in ordinary Arabic notation. The French encyclopedia of mathematics (t. 1: 3, p. 51) states that $2^{2^{36}}$ is an integer of more than twenty billion digits. The two following interesting propositions have been proposed, but no published proofs seem to exist: All the numbers represented in the series $2 + 1$, $2^2 + 1$, $2^{2^2} + 1$, $2^{2^{2^2}} + 1$, $\cdots$ are primes. The second is due to Eisenstein: There is an infinity[2] of prime integers of the form $2^{2^n} + 1$.

186. Proposed by H. PRIME, Boston, Mass.

Show that $\dfrac{(n+1)(n+2)\cdots(2n-2)}{(n-1)!}$ is an integer for all values of n.

SOLUTION BY THOMAS E. MASON, Bloomington, Ind.

The expression in the problem is equivalent to (1) $(2n-2)!/n!(n-1)!$ and the expression (2) $(m+n)!/m!n!$ is an integer, since it is a coefficient in the binomial expansion for the exponent $m + n$.

Each prime factor of the denominator of (1) which is not a factor of n occurs in the denominator as many times as it occurs in $(n-1)!(n-1)!$, and therefore occurs at least as many times in the numerator, since by (2) $(2n-2)!/[(n-1)!(n-1)!]$ is an integer.

Any prime factor of n (different from 1) is not contained in $n - 1$, and therefore occurs in the denominator of (1) as many times as it occurs in $n!(n-2)!$. It occurs at least as many times in the numerator since by (2) $(2n-2)!/n!(n-2)!$ is an integer.

The expression in the problem is therefore an integer for all values of n since the numerator contains every prime factor of the denominator at least as many times as it occurs in the denominator.

Also solved by B. F. YANNEY and H. C. FEEMSTER.

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[1] Intermed. Math., 10 (1903), p. 158; 11 (1904), p. 79. Lucas, Théorie des Nombres, p. 51.
[2] Lucas, Théorie des Nombres, p. 355.